Paper 2 Permutations, Combinations & Probability

Topic Overview
This section involves questions about arranging items, choosing items and probability itself. All of these ideas involve counting techniques, i.e. counting the number of possible outcomes when an experiment is performed. The trick is to analyse the question carefully and logically, before writing any numbers down on paper. Underline any important information in each question.

Key Concepts:

- **Fundamental Principle of Counting 1:** Stated simply, it is the idea that if we have *m* ways of doing something and *n* ways of doing another thing, then there are *m* X *n* ways of performing one operation followed by the other.

(First Operation) **AND** (Second Operation)

m X n

'And means multiply'

- **Fundamental Principle of Counting 2:** Stated simply, it is the idea that if we have *m* ways of doing something and *n* ways of doing another thing. Then the number of possible outcomes of the first operation (*m*) **OR** the second operation (*n*) is given by $n + m$.

(First Operation) **OR** (Second Operation)

m + n

'Or means add'

- **Permutations:** A permutation is an arrangement of a number of objects **in a definite order**.

- **Factorial *n*! : *n*!** is n(n-1)(n-2)……(3)(2)(1).

Example: 6! = 6x5x4x3x2x1 6! = 720
4! = 4x3x2x1 4! = 24

- **Combinations:** A combination is a selection of objects **in any order**.

To find a combination we use the '**nCr**' notation.

$^{n}C_{r} = \binom{n}{r}$ **An easy formula to remember this is:** $\binom{n}{r} = \dfrac{\textbf{First } r \textbf{ numbers counting down from } n \textbf{ to } 1}{\textbf{Last } r \textbf{ numbers in the countdown from } n \textbf{ to } 1}$

Example: $\binom{8}{3} = \dfrac{8x7x6}{3x2x1} = 56$

Probability: Probability is the measure of the chance, or the likelihood, of something happening.

The mesure of the probability of an event, E, is given by:

$$\mathbf{P}(E) = \frac{\textit{Number of successful outcomes}}{\textit{Number of possible outcomes}}$$

The probability of an event is a number between 0 and 1.

Note: **P(*E*)** = 0 means that an event is **impossible**.
P(*E*) = 1 means that an event is **certain**.

The Probability Scale: Is a scale from 0 to 1 which shows the probability of an event. The values can be given as fractions, decimals or percentages (0% to 100%).

Probability Scale:

The closer you move to 1 the more likely an event occurs. The closer to 0 the less likely.

Note: It is important to remember that the probability of all outcomes of an experiment will add up to 1.

Probability of an event not occurring = 1 – Probability of event occurring.
Example: Probability of not rolling a 5 with a fair dice = 1 – Probability of rolling a 5.

$$\text{P(not 5)} = 1 - \frac{1}{5}$$

$$\text{P(not 5)} = \frac{4}{5}$$

Probability Key Terms:

- **Trial:** Each time you carry out an experiment such as toss a coin, roll a dice, etc.
- **Outcomes:** The possible results that can occur.
- **Event:** Are the outcomes of interest.
- **Random:** Means equally likely to occur.
- **Unbiased:** Means fair.
- **Sample Space:** Is an ordered list of all possible outcomes:
  Example: Draw a sample space that shows the outcomes when two die are rolled and the outcomes are added together.

Die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

6 on one die and 6 on the other gives a

- **Relative Frequency and Experimental Probability:**
  Experiment probability involves carrying out an experiment, such as rolling a die 600 times and recording each result. The results of the experiment can be given as the experimental probability or the **Relative Frequency** of an event**.**

  Relative Frequency = $\frac{\boldsymbol{Number\ of\ succesful\ trials}}{\boldsymbol{Total\ number\ of\ trials}}$

  Example: A coin is tossed 400 times and there were 287 heads. Find the **relative frequency** of a head.

  Relative Frequency = $\frac{287}{400}$

  Note: if an experiment is repeated, that increasing the number of times an experiment is repeated generally leads to better estimates of probability.

- **Expected Frequency:** This relates to the expected results of an experiment. From the example above we would expect that if we tossed a coin 400 times the expected result should be 200 heads.

  Solution: 400 x P(Head)/ $\frac{1}{2}$ = 200

- **Addition Rule (OR Rule):**

> The probability that two events, A or B, can happen is given by:
>
> **P (*A or B*) = P(*A*) + P(*B*) – P(*A and B*)**
>
> **Or means Add**
>
> **Removes double counting**

This type of question can often occur if you are asked the probability using a list of numbers, for example finding the probability of a number divisible by 3 or a number divisible by 5 from the numbers 1 to 20.
As 15 is divisible by 5 and 3 this could be counted twice.
You could also solve this type of question by writing out all the possible numbers.

- **Mutually Exclusive Events:** The outcomes of two events that cannot happen at the same time. If events are mutually exclusive then you will not have to use the **OR Rule** a**s** there cannot be double counting.

- **Multiplication Rule (AND Rule):**

> The probability that two events, A and then B, both happen and in that order is given by:
>
> **P (*A* and *B*) = P(*A*) x P(*B*)**
>
> **where P(B) is worked out assuming that A has already occurred.**

> When the question says **and,** then **multiply.**

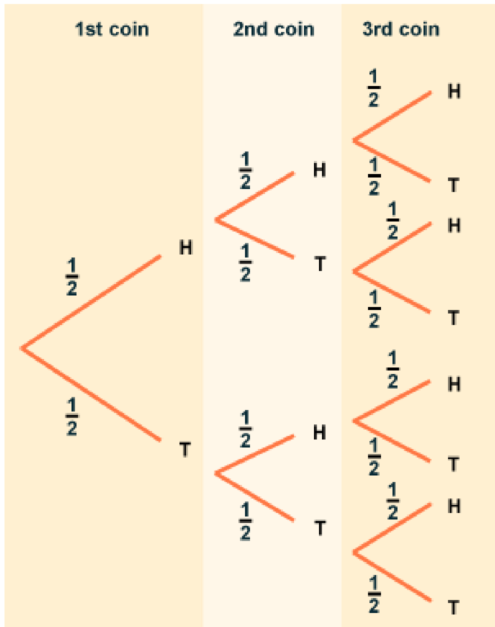
- **Bernoulli Trials:** Are trials that have only two possible outcomes: success or failure. For example rolling a dice to get exactly 5: all other numbers are a failure while only 5 is a success. P(Success) = $\frac{1}{6}$, P(Failure) = $\frac{5}{6}$.
  The result of each trial is independent; therefore the probability of success (or failure) does not change from one trial to another.
  We can be asked to answer questions involving up to **three** Bernoulli Trials.
  Example of a question involving Bernoulli Trials: If a success in a game is rolling a five on a dice. Calculate the probability that the **first** success occurs on the **third** trial.
  Solution: P(Success) = $\frac{1}{6}$, P(Failure) = $\frac{5}{6}$

1st Trial		***2nd Trial***		***3rd Trial***
Failure	**and**	Failure	**and**	Success
$\frac{5}{6}$	X	$\frac{5}{6}$	X	$\frac{1}{6}$

Answer: $\frac{25}{216}$

- **Tree Diagram showing probability:**

Tree diagrams allow us to see all the possible outcomes of an event and calculate their probability. Each branch in a tree diagram represents a possible outcome.
If two events are **independent**, the outcome of one has no effect on the outcome of the other. For example, if we toss two coins, getting heads with the first coin will not affect the probability of getting heads with the second.
A tree diagram which represent a coin being tossed three times looks like this diagram.
This type of diagram could also be used to solve a **Bernoulli Trial**.



- **Expected Value/E(X):** This is not the expected frequency. The expected value is the average outcome of an experiment.
  To calculate the expected value, we multiply every possible outcome by the probability for that event occurring and then add these values together.
  Formula: Expected Value/E(X) = Sum of each outcome multiplied by its probability.

  Note: The expected value does not have to be an outcome.

Expected Value Example: Find the expected value of rolling an unbiased dice:

Outcome		Probability		
1	x	$\frac{1}{6}$	=	$\frac{1}{6}$
2	x	$\frac{1}{6}$	=	$\frac{2}{6}$
3	x	$\frac{1}{6}$	=	$\frac{3}{6}$
4	x	$\frac{1}{6}$	=	$\frac{4}{6}$
5	x	$\frac{1}{6}$	=	$\frac{5}{6}$
6	x	$\frac{1}{6}$	=	$\frac{6}{6}$

Adding these values together gives the expected value:

$$E(X) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}$$

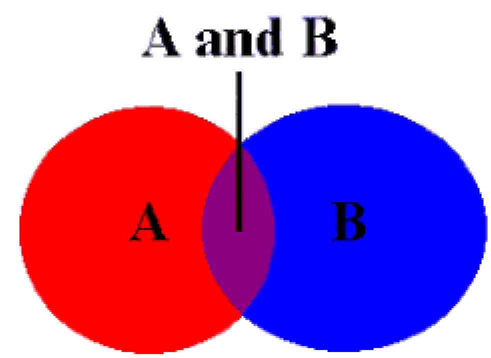
E(X) = 3.5 (Calc)

Consider playing a game where each roll of the die paid out that value in euro (for example a 3 pays €3). We would expect to win **on average** €3.50

The expected value can be used to determine whether an experiment is fair or not and whether a bet is good or bad value.
In general:

- If the expected value > 0, we would expect to gain that amount on average.
- If the expected value = 0, the game is fair, we are equally likely to win our lose.
- If the expected value < 0, we would expect to lose that amount on average.

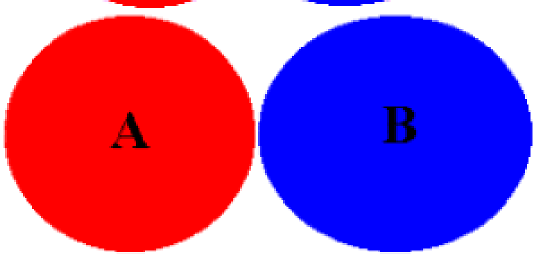
➢ **Venn Diagrams and Probability:** We can also use Venn diagrams to calculate probability.



A and B can occur together.

N.B. The probability of A and B is written as:
P(A and B) OR P(A ∩ B)

The probability of A or B can be written as:
P(A or B) OR P(A ∪ B)



A and B are mutually exclusive. A and B cannot occur together.

Reminder of some of the Set Symbols:
$\cap$: $\boldsymbol{Intersection}$ $\cup$: $\boldsymbol{Union}$ $\boldsymbol{A'}$: $\boldsymbol{A\ compliment\ (all\ values\ outside\ set\ A)}$
A\B: A less B (set A take away any values in set B)

Below is a least of the types of questions that were asked in the old syllabus. These questions are good examples of how questions in this section can be asked.

1. **Calculator Work**
2. **Arrangements**
3. **Selections/Combinations**
4. **Probability**

1. Calculator Work.

You can be asked questions based on using your calculator. You should be familiar with the factorial button: ***!*** and the combination button: ${}^{n}C_{r}$ on your calculator.
This part can take seconds if you know which calculator button to use.

2. Arrangements (this is often called Permutations)

In this type of question you are asked to arrange objects in a particular order.
To answer an Arrangement question use the box method that we used in class.
If there is restriction, as in the question below, you must fill in the restrictions first.

2004 Paper 2

(a) The letters of the word CUSTOMER are arranged at random.

(i) How many different arrangements are possible?

(ii) How many of these arrangements begin with the letter C?

Q6(a)(i)

There are 8 ways to fill the first box. Once this is filled, there are 7 ways to fill the second box and so on.

Number of ways = $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320$

6 (a) (ii)

There is only one way to fill the first box (with the letter C). Once this is filled, there are 7 ways to fill the second box and so on.

Number of ways = $1 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040$

$\boxed{C}$

2006 Paper 2 Q6

(c) (i) How many different three-digit numbers can be formed from the digits 2, 3, 4, 5, 6, if each of the digits can be used only once in each number?

(ii) How many of the numbers are less than 400?

(iii) How many of the numbers are divisible by 5?

(iv) How many of the numbers are less than 400 and divisible by 5?

The 1st. box can be filled by any one of 5 digits.
AND then the 2nd. box can only be filled by one of 4 digits, given that the 1st. box is already filled by a digit.
AND then the 3rd. box can be only filled by one of 3 digits.
Now multiply the three numbers together to get 60 arrangements.

Number of ways $= 5 \times 4 \times 3 \quad = 60$

$\boxed{}\,\boxed{}\,\boxed{}$

6 (c) (ii)

The first box can only be filled 2 ways, by either a 2 or 3. Any other digits will give a number greater than 400. Once this is filled, there are 4 ways to fill the second box and 3 ways to fill the third box.

Number of ways $= 2 \times 4 \times 3 \quad = 24$

$\boxed{}\,\boxed{}\,\boxed{}$

Must be a 2 or 3

6 (c) (iii)

The last digit must be the 5 in order to be divisible by five. Therefore, there is only 1 way to fill the last box. This means there are 4 ways to fill the first box and 3 ways to fill the second box.

Number of ways $= 4 \times 3 \times 1 \quad = 12$

$\boxed{}\,\boxed{}\,\boxed{5}$

6 (c) (iv)

The first box must be filled by a 2 or 3 (two ways) in order to have a number less than 400.
The last box must be filled by a 5 (one way) in order to be divisible by 5.
This means there are 3 ways to fill the second box.

Number of ways $= 2 \times 3 \times 1 \quad = 6$

$\boxed{}\,\boxed{}\,\boxed{5}$

Must be a 2 or 3

In 2009 Paper 2 Q6 (c) (iv) they asked a question that was a little more difficult:
Three boys and two girls are seated in a row as a group. In how many different ways can the group be seated if (iv) the two girls must be seated beside each other?
In this case you must count the girls as one person. Which should give 4x3x2x1.
<u>However</u> as the two girls can be arranged in two different ways the solution is 4x3x2x1x2x1=48

3. Selections (Combinations)

A selection of objects from a given set, without regard to order is called a **combination**.

In combinations we use the following:

The number of selections of n different objects taking r at a time $= {}^{n}C_{r} = \binom{n}{r}$

Again you should be familiar with the correct button on your own calculator.

Often you will be asked to select from two different groups. There are two key words that apply in these cases:

- **'And'** is understood to mean multiply. Thus, **and = x**
- **'Or'** is understood to mean add. Thus, **or = +**

2004 Paper 1

(b) A committee of 3 people is selected from a group of 15 doctors and 12 dentists.

In how many different ways can the 3 people be selected

(i) if there are no restrictions

(ii) if the selection must contain exactly 2 doctors

(iii) if the selection must contain at least 1 doctor and at least 1 dentist

(iv) if the selection must contain one specific doctor and one specific dentist?

(i) The number of ways of selecting 3 people from 27 people is: $^{27}C_3 = \binom{27}{3} = \frac{27 \times 26 \times 25}{3 \times 2 \times 1} = 2{,}925$

15 Doctors 12 Dentists
↓
3 Places

CALCULATOR: Calculate $^{27}C_3$.

27 SHIFT nCr 3 =

Math
27C3
2,925

(ii) You need to select 2 doctors from 15 doctors **AND** 1 dentist from 12 dentists.

15 Doctors 12 Dentists
↓ ↓
2 Places 1 Place

$^{15}C_2 \times {}^{12}C_1 = \left(\frac{15 \times 14}{2 \times 1}\right) \times \left(\frac{12}{1}\right) = 1{,}260$

Use Calculator button ^nC_r

NOTE: **AND** means you multiply.

6 (b) (iii)

At least one doctor and at least one dentist can mean 2 doctors and 1 dentist **OR** 1 doctor and 2 dentists.

15 Doctors 12 Dentists
↓ ↓
2 Places 1 Place

OR

15 Doctors 12 Dentists
↓ ↓
1 Place 2 Places

2 doctors and 1 dentist: $^{15}C_2 \times {}^{12}C_1 = 1{,}260$

1 dentist and 2 doctors: $^{15}C_1 \times {}^{12}C_2 = \left(\frac{15}{1}\right) \times \left(\frac{12 \times 11}{2 \times 1}\right) = 990$

NOTE: **OR** means you add.

2 doctors and 1 dentist **OR** 1 dentist and 2 doctors = 1,260 + 990 = 2,250

6 (b) (iv)

If one specific doctor is chosen and one specific dentist is chosen you are left to pick one person from 14 doctors and 11 dentists (25 people).

14 Doctors 11 Dentists
↓
1 Place

$^{25}C_1 = \left(\frac{25}{1}\right) = 25$

2008 Paper 2 Q6

(c) There are 6 junior-cycle students and 5 senior-cycle students on the student council in a particular school.
A committee of 4 students is to be selected from the students on the council.

In how many different ways can the committee be selected if

(i) there are no restrictions

(ii) a particular student must be on the committee

(iii) the committee must consist of 2 junior-cycle students and 2 senior-cycle students.

6 (c) (i)

In how many ways can 4 people be selected from 11 people?

11 students
↓
4 Places

$$^{11}C_4 = \frac{11\times10\times9\times8}{4\times3\times2\times1} = 330$$

CALCULATOR: Calculate $^{11}C_4$.

11 SHIFT **nCr** **4** **=**

11C4

330

6 (c) (ii)

If one student is already on the committee you need to find out in how many ways 3 people can be selected from 10 people?

10 students
↓
3 Places

$$^{10}C_3 = \frac{10\times9\times8}{3\times2\times1} = 120$$

6 (c) (iii)

In how many ways can you pick 2 junior-cycle students from 6 junior-cycle students AND 2 senior-cycle students from 5 senior-cycle students?

6J	5S
↓	↓
2 Places	2 Places

$$^{6}C_2 \times {}^{5}C_2 = \frac{6\times5}{2\times1}\times\frac{5\times4}{2\times1} = 150$$

NOTE: AND means multiply.

4. Probability Rules:

The probability rules were discussed above.

$$P(E) = \frac{\text{number of favourable outcomes in } E}{\text{number of possible outcomes}},$$

where $0 \le P(E) \le 1$.

The Addition Rule

$P(A \textbf{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

This rule stops double counting, this often occurs in questions similar to those that ask for the probability of a multiple of 3 or the probability of a multiple of 5 occuring.

The Multiplication Rule

$P(A \textbf{ and } B) = P(A) \times P(B)$

This is the probability that two events happen, eg a coin is tossed and a dice rolled.

Note:

One operation **and** a second operation ⇒ **multiplication**

One operation **or** a second operation ⇒ **addition**

Probability of Event not happening

$P(\text{not } E) = 1 - P(E)$

$P(\text{certainty}) = 1$

$P(\text{impossibility}) = 0$

2008 Paper 2 Q6

(b) One shelf of a school library has 70 books. The books are on poetry and on drama and are either hardback or paperback.
The following table shows the number of each type.

	Hardback	Paperback
Poetry	23	17
Drama	14	16

A student selects one book at random from the shelf.

Find the probability that the book selected is

(i) a paperback poetry book

(ii) a hardback book

(iii) a poetry book

(iv) not a paperback dra

Example: If a die is rolled what is the probability of the outcome **not** being a six.
P(of a 6) = 1/6

P(of not a 6) = 1 – 1/6 = 5/6

6 (b)

Poetry		
Drama	14	16

Total number of books = 70
Total number of poetry books = 40
Total number of drama books = 30
Total number of hardback books = 37
Total number of paperback books = 33

$$p(E) = \frac{\text{Number of desired outcomes}}{\text{Total possible number of outcomes}}$$ …… 4

6 (b) (i)

$$p(\text{Picking a paperback poetry book}) = \frac{\text{No. of paperback poetry books}}{\text{No. of books}} = \frac{17}{70}$$

6 (b) (ii)

$$p(\text{Picking a hardback book}) = \frac{\text{No. of hardback books}}{\text{No. of books}} = \frac{37}{70}$$

6 (b) (iii)

$$p(\text{Picking a poetry book}) = \frac{\text{No. of poetry books}}{\text{No. of books}} = \frac{40}{70} = \frac{4}{7}$$

6 (b) (iv)

$$p(\text{Picking a book that is not a paperback drama book})$$

$$= \frac{\text{No. of non paperback drama books}}{\text{No. of books}} = \frac{54}{70} = \frac{27}{35}$$

Note: There are variations of this question that can cause problems:

If a drama book is picked at random what is the probability that it is a paperback?

$$\frac{\textbf{No.of paperback drama books}}{\textbf{No.of drama books}} = \frac{\mathbf{16}}{\mathbf{30}} = \frac{\mathbf{8}}{\mathbf{15}}$$

Project Maths Probability Syllabus:

Students learn about	Students working at FL should be able to	In addition, students working at OL should be able to
1.1 Counting	– list outcomes of an experiment – apply the fundamental principle of counting	– count the arrangements of n distinct objects ($n!$) – count the number of ways of arranging r objects from n distinct objects
1.2 Concepts of probability	– decide whether an everyday event is likely or unlikely to happen – recognise that probability is a measure on a scale of 0-1 of how likely an event is to occur – connect with set theory; discuss experiments, outcomes, sample spaces – use the language of probability to discuss events, including those with equally likely outcomes – estimate probabilities from experimental data – recognise that, if an experiment is repeated, there will be different outcomes and that increasing the number of times an experiment is repeated generally leads to better estimates of probability – associate the probability of an event with its long run relative frequency	– discuss basic rules of probability (AND/ OR, mutually exclusive) through the use of Venn Diagrams – calculate expected value and understand that this does not need to be one of the outcomes – recognise the role of expected value in decision making and explore the issue of fair games

Students learn about	Students working at FL should be able to	In addition, students working at OL should be able to
1.2 Concepts of probability	– recognise that, if an experiment is repeated, there will be different outcomes and that increasing the number of times an experiment is repeated generally leads to better estimates of probability – associate the probability of an event with its long run relative frequency	
1.3 Outcomes of random processes	– construct sample spaces to show all possible outcomes for two independent events – apply the principle that in the case of equally likely outcomes the probability is given by the number of outcomes of interest divided by the total number of outcomes (examples using coins, dice , spinners, urns with coloured objects, playing cards etc.)	– find the probability that two independent events both occur – apply an understanding of Bernoulli trials* – solve problems involving up to 3 Bernoulli trials – calculate the probability that the 1st success occurs on the n^{th} Bernoulli trial where n is specified